Ph.D. Comprehensive Exam

Department of Physics
Georgetown University

Part I: Monday, August 25, 2008, 2:00pm - 6:00pm

Instructions:

- This is a closed-book, closed-notes exam. Calculators are not allowed.
- Each problem is worth 20 points.
- You should submit work for all of the problems. Note that in some cases, even if you get stuck on one part of a problem, you may be able to make progress on subsequent parts.
- If you need additional paper to work the problems, please use separate sheets for different problems.
- Show all your work.

Name:

1. A particular quantum system has two different energy levels, the ground state energy $E_0 = 0$, and excited state energy E_i. The ground state is unique, *i.e.*, has a degeneracy of one, and the excited state has a degeneracy of n, *i.e.*, there are n distinguishable states of energy E_i. The system is in equilibrium at temperature T.

(a) Derive an expression for the free energy of the system.

(b) Calculate the probability that the excited state will be occupied.

(c) Calculate the average energy of the system.

(d) Calculate the entropy of the system. Determine the entropy as $T \to 0$ and $T \to 1$. Explain why these results are reasonable.

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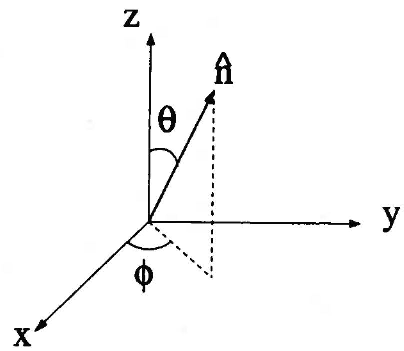
2. Consider an electron at rest in a uniform magnetic field, $\vec{B} = B_0\hat{z}$. The Hamiltonian of the system is given by

$$H = -\vec{\mu}\cdot\vec{B},$$

where $\vec{\mu} = -\frac{e\hbar}{2m}\vec{\sigma}$.
The Pauli spin matrices are given by

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1}$$

in the $\{|+>, |->\}$ basis of σ_z eigenstates.
At time $t = 0$, the electron spin is pointing along the $+\hat{n}$ direction, as shown in the figure.



(a) Determine the matrix representing $\sigma_{\hat{n}} = \vec{\sigma}\cdot\hat{n}$ in the basis of σ_z eigenstates.

(b) Show that the initial state of the electron spin can be represented by

$$\chi(0) = \begin{bmatrix} \cos\frac{\theta}{2}\exp^{-i\phi/2} \\ \sin\frac{\theta}{2}\exp^{i\phi/2} \end{bmatrix} \tag{2}$$

in the σ_z eigenbasis.

(c) Find $\chi(t)$.

Some trig. identities:

$\sin 2\theta = 2\sin\theta\cos\theta$
$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$\sin^2(\theta/2) = (1-\cos\theta)/2$
$\cos^2(\theta/2) = (1+\cos\theta)/2$

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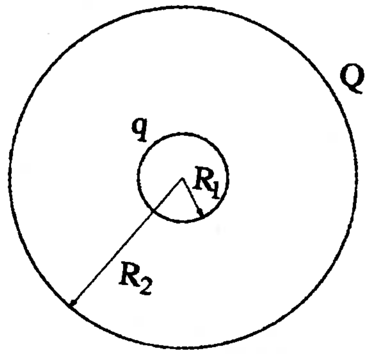
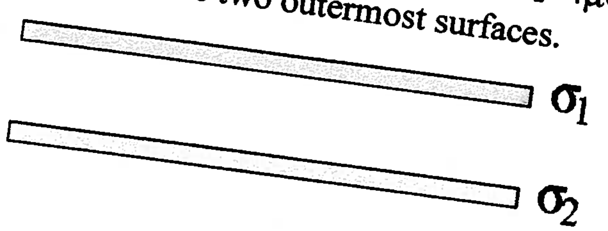
3.

a) A schematic description of a van de Graaff generator for producing large voltages is that of two concentric, metal spheres. Charge delivered to the inner sphere by a system of brushes and moving belts is allowed to flow to the outer sphere by a wire with some resistance that connects the two spheres. If a charge of q is maintained on the inner sphere, explain with a short calculation why the accumulation of charge Q on the outer sphere does *not* reduce its potential difference with the inner sphere, making it harder to build up a large Q on the outside.



b) Assume now that the outer sphere is all charged up, with total charge Q, and that there is no charge on the inner sphere. The radius of the outer sphere is R_2. What is the energy stored in the electric field?

c) Consider two infinite, flat metallic plates, both parallel to the *xy*-plane, with the *z*-direction vertical. Each plate is 0.5 cm thick. The gap between the plates is 2 cm. The top plate has a net surface charge density $\sigma_1 = 3\mu C/m^2$ (including both surfaces), and the bottom plate has a net charge of $\sigma_2 = 4\mu C/m^2$. Find the surface densities σ_{top} and σ_{bottom} on the two outermost surfaces.



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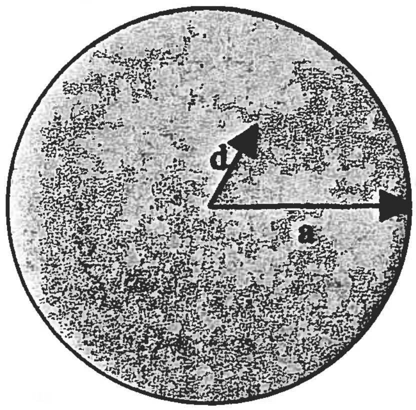
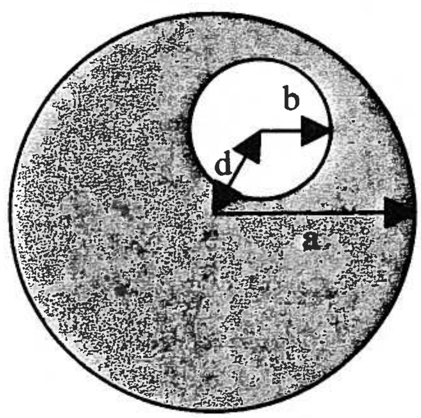
4. An infinite metal cylinder has radius a, as shown in the figure below. A current I (with uniform current density) flows in the cylinder along the axis. The direction of the current is into the page.



(a) What is the magnitude and direction of the B-field due to the current at the point indicated in the figure, a distance d from the cylinder axis?

Now a cylindrical hole is bored all the way through a metal cylinder parallel to the cylinder's axis. The hole has radius b and is offset from the center of the cylinder by a distance d, with d>b. The same current I flows into the page in the remaining part of the cylinder.



(b) What is the magnitude and direction of the B-field at the center of the <u>hole</u> due to this current?

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Part II: Tuesday, August 26, 2008, 2:00pm - 6:00pm

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- Show all your work.

Name: ______________________________

1. Consider a quantum harmonic oscillator with frequency ω_0.

   (a) Calculate the average value for the quantum number n when the system is in thermal equilibrium at temperature T, and show that the result is equivalent to the Bose-Einstein distribution function evaluated at energy $\hbar\omega_0$ with no chemical potential.

   (b) Determine the average energy of the system at temperature T.

Now consider a crystalline solid that has a phonon density of states $g(\omega)$ (per unit cell).

   (c) Write an expression for the phonon contribution to the energy of the solid at temperature T.

   (d) If an Einstein model is used to describe an optical phonon branch in the solid, with $\omega(\vec{k}) = \omega_0$, what is the phonon density of states $g(\omega)$ for that branch?

   (e) Name an experimental method for probing the vibrational spectrum of a solid and give a brief description of how the method works.

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2. A system in thermal equilibrium has an average energy given by $\langle E \rangle$.

   (a) Prove that the mean square deviation of the energy from $\langle E \rangle$, $\langle (E - \langle E \rangle)^2 \rangle$ is given by the following relationship:

$$\langle (E - \langle E \rangle)^2 \rangle = k_B T^2 C_v,$$

   where k_B is Boltzmann's constant, T is the system temperature and C_v is the heat capacity of the system at constant volume.

   (b) Using this result, show that the energy of a macroscopic system can be considered constant when the system is in thermal equilibrium.

Hints:
(a) Expand the left hand side of the equation and think about the average energy in terms of the partition function.
(b) Consider the fractional deviation of the energy of a macroscopic system. For this part you may assume that $\langle E \rangle = N k_B T$ (i.e., high temperature).

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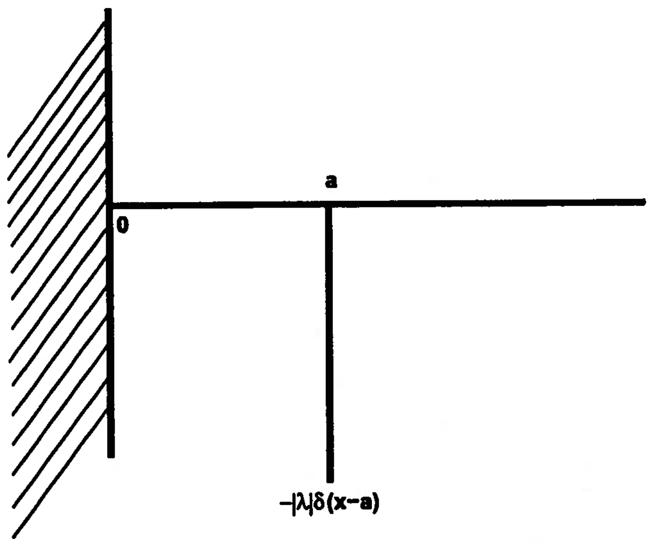

3. Consider a quantum particle moving in one dimension in a box with an infinite potential barrier at $x = 0$ and an attractive delta-function potential at $x = a$ [$V(x) = \infty$ for $x < 0$ and $V(x) = -|\lambda|\delta(x-a)$ for $x > 0$].

(a) This potential has only one bound state, regardless of the size of λ. By matching the appropriate boundary conditions (for an appropriate ansatz for the wavefunction) find a transcendental equation that determines the energy eigenvalue for this bound state.

(b) Use graphical methods to solve the transcendental equation. What limiting value does the energy take for large $|\lambda|$?

(c) By first normalizing the wavefunction, find the probability that the particle is found in the region $x > a$.

Some possibly useful relations:

$\cosh x = (e^x + e^{-x})/2$
$\sinh x = (e^x - e^{-x})/2$
$\tanh x = \sinh x / \cosh x$

Name:

4. Within the context of the Drude model, the average motion of electrons in a metal is governed by the equation

$$m\frac{d\vec{v}(t)}{dt} = -m\frac{\vec{v}(t)}{\tau} + \vec{F}_{ext}(t).$$

Here $\vec{v}(t)$ is to be understood as the electron velocity averaged over all the electrons at time t. Of course, m is the electron mass, t denotes time, τ is the relaxation time, which is taken to be constant, and $\vec{F}_{ext}(t)$ is the average external force on the electrons at time t.
Use the Drude model to calculate the ac electrical conductivity, $\sigma(\omega)$, i.e., the response to an electric field of the form

$$\vec{E}(t) = Re[\vec{E}(\omega)e^{-i\omega t}],$$

where $\vec{E}(\omega)$ is a vector constant in space and ω is the angular frequency of the electric field.
You are, of course, interested in the response for times t such that $t >> \tau$, i.e., the steady state response.
You should obtain an electrical current density of the form

$$\vec{J}(t) = Re[\sigma(\omega)\vec{E}(\omega)e^{-i\omega t}].$$

Express your result for $\sigma(\omega)$ in terms of the dc electrical conductivity and quantities defined in the statement of this problem.
Be careful to define and explain any additional symbols that you use.