Ph.D. Comprehensive Exam

Department of Physics
Georgetown University

Part I: Monday, August 30, 2010, 1:00pm - 5:00pm

Instructions:

- This is a closed-book, closed-notes exam. The only electronic devices allowed are calculators provided by the department.
- Each problem is worth 40 points.
- You should submit work for all of the problems. Note that in some cases, even if you get stuck on one part of a problem, you may be able to make progress on subsequent parts.
- Please work different problems on separate sheets of paper.
- Show all your work.

Name: _____

1. A particle moving in one dimension is confined to an infinite square well potential extending from $x = 0$ to $x = L$, as shown in figure 1. At $t = 0$, the particle's wave function, $\Psi(x, 0)$, is piecewise linear and symmetric about $x = L/2$, with a maximum height of A, as shown in figure 2.

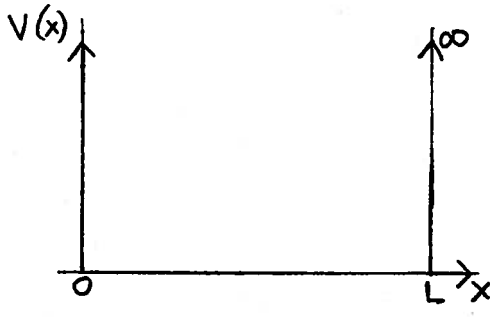


Fig. 1

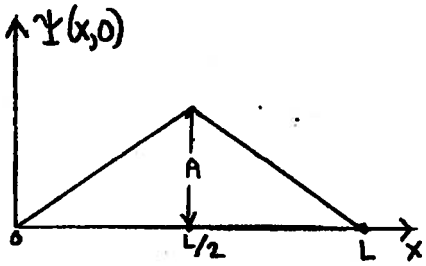


Fig. 2

(a) Find A.

(b) Write down the normalized energy eigenfunctions, $\phi_n(x)$, for a particle in this potential, along with the corresponding energy eigenvalues, E_n. What are the allowed values of n?

(c) Consider an expansion of the particle's wave function $\Psi(x,0)$ in terms of the eigenfunctions found in part (b):

i. Without doing any calculations, you should be able to see that half of the eigenfunctions do not appear in the expansion of $\Psi(x,0)$. Explain why. (*Hint: it might help to sketch some of the eigenfunctions.*)
ii. Determine the expansion coefficients for the other half of the eigenfunctions.

(d) Write an expression for $\Psi(x, t)$. You may leave your answer in terms of the expansion coefficients calculated in part (c).

(e) How much time, T, must elapse for the wave function to repeat itself, i.e. find the smallest $T>0$ such that $\Psi(x,T) = \Psi(x,0)$.

(f) What is the expectation value of the particle's energy at an arbitrary time t? You may leave your answer in terms of the expansion coefficients calculated in part (c).

Name:

2. The Hamiltonian of a one-dimensional harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

Recall that if the system is a quantum system, the energy eigenvalues are

$$E_n = \hbar\omega(n + \frac{1}{2}),$$

where n is a nonnegative integer, and the energy eigenfunctions in the position representation are

$$u_n(x) \equiv \langle x|n\rangle = \left[\frac{C}{\pi^{1/2}2^n n!}\right]^{1/2} \exp\left(\frac{-C^2x^2}{2}\right) H_n(Cx),$$

where H_n are Hermite polynomials, $C = \sqrt{\frac{m\omega}{\hbar}}$, and $|n\rangle$ is the corresponding state vector.

(a) What is the differential equation satisfied by $u_n(x)$?

(b) What is the differential equation satisfied by $\bar{u}_n(p) \equiv \langle p|n\rangle$ (i.e., the energy eigenfunctions in the momentum representation)?

(c) By comparing your answers to (a) and (b), determine $\bar{u}_n(p)$.

(d) What is the relationship between $u_n(x)$ and $\bar{u}_n(p)$?

(e) At time t_0, the state vector of the system is

$$|\psi_0\rangle = \frac{i}{2}|0\rangle - \frac{\sqrt{3}}{2}|2\rangle.$$

What is the state vector of the system at some later time t_1?

(f) At time t_1, what is the probability that the system is in the ground state?

(g) Suppose at time t_1 the energy of the system is measured to be $E = \hbar\omega/2$. At some later time t_2, what is the probability that the system will be in the ground state?

Name:

3.

A P-N junction device is shown below. The junction is centered at x=0 and the potential in the device is given by the function:

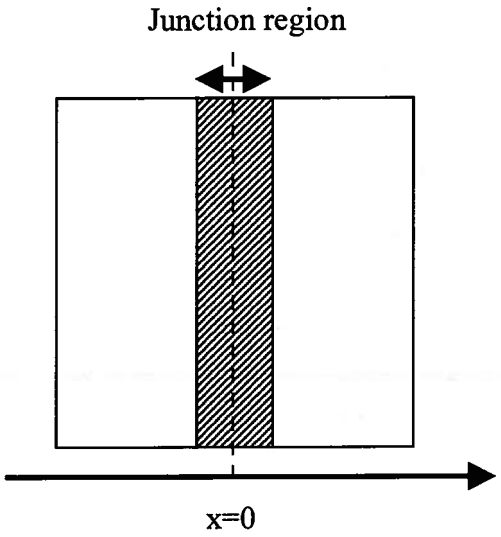
$$\phi(x)=\left(1+\frac{x}{\sqrt{x^2+a^2}}\right)\cdot\frac{V_0}{2}$$

where a is a measure of the width of the junction. In the approximation that the width of the device is much greater than the width of the junction region, V_0 is the total potential difference across the device.

a) Find the electric field E and the charge density ρ as a function of x. Specify which set of units you are using (i.e., SI, Gaussian, etc.).

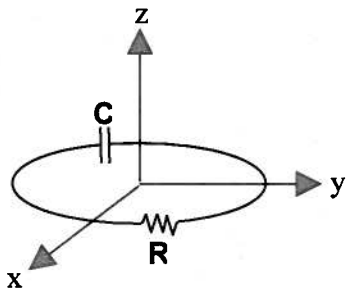
b) Plot the functions φ(x), E(x), and ρ(x) as functions of x. Label the axes so that the maxima or minima of the functions can be ascertained.

c) Find the net charge on the device.



Name:

4. A conducting wire, placed in a magnetic field, $\mathbf{B} = B_0 \sin(\omega t)\,\hat{z}$, forms a circular loop and lies in the $z = 0$ plane with a radius of 0.100 m (see figure). The magnitude of the field is $B_0 = 0.200$ Tesla, and the angular frequency is $\omega = 1.00 \times 10^3$ rad/s. The wire is series connected to a 5.00 Ω resistor and a 1.00×10^2 μF capacitor, as shown. Determine the current in the conducting wire.



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Name: _____

1.(a) A monoatomic ideal gas is adsorbed to a two dimensional surface. At a high temperature T, what is the heat capacity per mole of molecules on the surface?

(b) Now imagine that the density of the 2D ideal gas is raised until the particles begin to weakly interact with each other, resulting in a two dimensional solid. The Cartesian components of the force that confines each particle to the neighborhood of its equilibrium position are proportional to the cubic power of the displacement in that direction (*i.e.* $F_x \propto x^3$ and $F_y \propto y^3$). Calculate the heat capacity per mole of this two-dimensional solid at a high temperature T.

Hints:
Remember that there are N_A, Avogadro's number, molecules per mole and that high temperature means you can treat the system classically.
If you are stuck, it might help to write down expressions for the energy of a single particle in the system, and the ensemble average of the energy of this classical system.

Name:

2. Consider a one-dimensional and quantum-mechanical system of two spin s particles of mass m for which the system Hamiltonian is

$$H = \sum_{K=1}^{2} \left(\frac{p_K^2}{2m} + \frac{1}{2} m\omega^2 x_K^2 \right).$$

Here x_K and p_K are the position and linear momentum, respectively, of the k th particle and ω is a positive angular frequency. Of course, s is a non-negative integer for bosons and a positive half-odd-integer for fermions. The z-component of a particle's spin can take on the values (or better, has the eigenvalues) $-\hbar s, \ldots, \hbar(s-1), \hbar s$.

a. **What are the allowed single-particle energies?**

b. **What are the allowed energies for the system of two particles?**

c. Consider the system when it is in equilibrium at an absolute temperature given by

$$T = \frac{1}{k_B \beta},$$

where k_B is the Boltzmann constant.

(1) **Calculate the canonical ensemble partition function if the particles can be treated as distinguishable spin 0 particles.** (Note that this can be arranged in at least two ways: (i) the particles are spin 0 bosons and T is sufficiently large; (ii) the particles are *manufactured* nanometer-scale spheres.)

(2) **Calculate the canonical ensemble partition function if the particles are identical and thus indistinguishable bosons,**

(3) **Calculate the canonical ensemble partition function if the particles are identical and thus indistinguishable fermions.**

HINT: THE SERIES ENCOUNTERED ARE GEOMETRIC SERIES.

Name:

3. Consider a wire of length $L = 1.00$ m and cross-sectional area $A = 4.00$ mm^2. One end of the wire is attached to the ceiling. You hang a 10.0 kg mass from the bottom of the wire and measure that the wire stretches by $\Delta L = 0.750$ mm. In this problem, you will use this measurement to estimate the spring constant of the interatomic bonds and a typical phonon frequency in this material.

(a) Treat the wire as a macroscopic ideal spring.

- Determine the spring constant k_{wire} of the wire.

(b) The wire is actually made up of lots of individual atoms. We will adopt a microscopic ball-and-spring model for the wire: think of the wire as consisting of many long chains in parallel. Each chain consists of atoms (balls) connected by interatomic bonds (springs), as shown in the figure.[1] (For clarity, the horizontal bonds are not shown in the figure; we will assume that they do not change length when the wire is stretched.)

- What is a typical distance d between neighboring atoms in a solid?
- Use your estimate for d to estimate the number of parallel atomic chains in the wire (N_{chains}) and the number of atoms within each chain (N_{links}). (You may assume a simple cubic crystal structure.)

(c) Recall that for two identical springs with spring constant k_0 in series, the effective spring constant is $k_{ser} = k_0/2$. For two identical springs in parallel, the effective spring constant is $k_{par} = 2k_0$.

- Estimate the spring constant k_{bond} for an individual interatomic bond in the wire.

(d) Let $u(r)$ be the potential energy arising from the interaction between two atoms separated by distance r.

- Make a sketch of a generic interatomic potential energy curve $u(r)$.
- In what way is the interatomic spring constant k_{bond} related to features in your graph? Derive a mathematical expression that relates k_{bond} to properties of the function $u(r)$. (Do not assume a specific functional form for $u(r)$.)

(e) Assume the wire is made out of aluminum. The atomic mass of Al is 27.0 amu, and 1 amu $\approx 1.67 \times 10^{-27}$ kg. (Don't worry about the fact that the crystal structure of Al is actually not simple cubic.)

- Use your value for k_{bond} to estimate a typical phonon frequency, ω, for this material.

[1] Figure taken from *Matter and Interactions I: Modern Mechanics*, R. Chabay and B. Sherwood (Wiley, 2007).

Name:

4. Consider a macroscopic, two-dimensional system of N non-interacting, spin-$\frac{1}{2}$ Fermions with mass m and energy spectrum

$$\epsilon(\vec{k}) = \epsilon_0 + \frac{\hbar^2 k^2}{2m},$$

in an area A. Here ϵ_0 is a constant.

(a) Derive an expression for k_F, the magnitude of the Fermi wave vector, in terms of the particle density $n = N/A$.

(b) Determine $D(\epsilon)$, the density of states as a function of energy.

(c) Calculate $\mu(T)$, the chemical potential of the Fermions, when they are in equilibrium at an absolute temperature T.