Ph.D. Comprehensive Exam

Department of Physics
Georgetown University

Part I: Monday, August 29, 2011, 1:00pm - 5:00pm

Instructions:

- This is a closed-book, closed-notes exam. The only electronic devices allowed are calculators provided by the department.
- Each problem is worth 50 points.
- You should submit work for all of the problems. In many cases, even if you get stuck on one part of a problem, you may be able to make progress on subsequent parts.
- Please work different problems on separate sheets of paper.
- Show all your work.

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1. Suppose that in the absence of any matter and charges, an electric field exists of the form

$$\vec{E} = E_0 \hat{x} \cos(kz - \omega t) + E_0 \hat{y} \sin(kz - \omega t).$$

In this problem, you will show that $\vec{E}$ satisfies Maxwell's equations, provided that a certain magnetic field $\vec{B}(x, y, z, t)$ also exists and a certain relation between k and ω is satisfied.

(a) What is the appropriate relation between k and ω?

(b) What is $\vec{B}(x, y, z, t)$?

(c) Show directly that this $\vec{E}$ and $\vec{B}$ satisfy all of Maxwell's equations.

(d) Describe what the electric and magnetic fields look like at the origin as a function of time.

(e) Determine the energy flow (both magnitude and direction).

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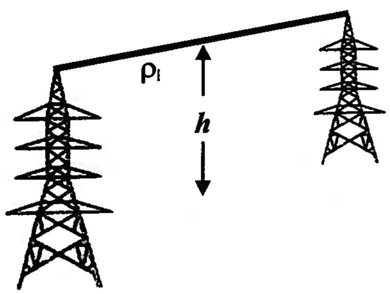
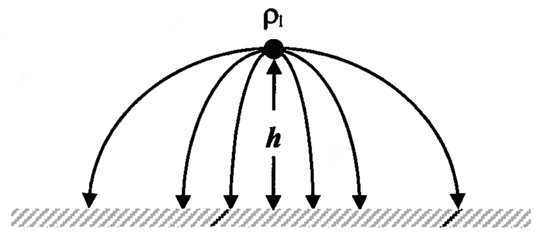
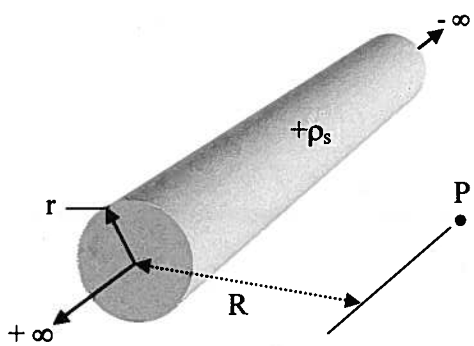
2. Consider an infinite line of charge with uniform charge per unit length, ρ_l, situated at a height of h above and parallel to the surface of an infinite, perfectly conducting ground plane, as shown in the figure below. The physical significance of such a configuration resembles a power transmission line above the earth (even if the earth is not a perfect conductor and the line is not infinite in length). Find the capacitance c per unit of axial length between the line and the ground plane using the method of images. You will arrive at the solution through a series of preliminary steps.



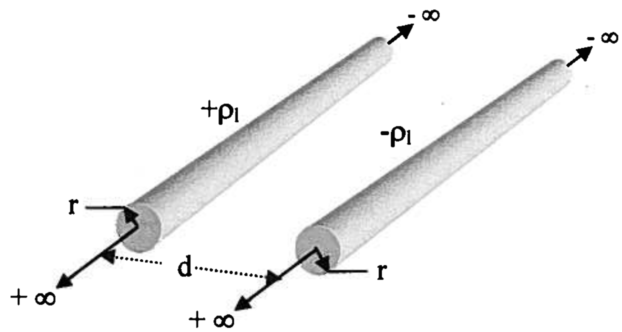


a) First determine the magnitude and direction of electric field, **E**, at any point P produced by a uniform surface charge distribution, ρ_s, situated on a cylindrical surface with radius r and having infinite length. Assume that point P is situated an arbitrary distance R from the axis of the cylinder and that $R > r$.



b) Determine the potential difference between any two points at radial distances R_a and R_b from the cylinder, where $R_b > R_a > r$.

c) Now consider two <u>identical</u> cylindrical conductors, infinite in length with radius r, one having a **line** charge distribution, $+\rho_l$, and the other having $-\rho_l$. Making use of the result from (b), calculate the capacitance per unit length, *c*, for the structure shown below. Assume that d >> r such that proximity effects are negligible, that is, the charge distribution on the cylinders remains uniform.



d) Finally, back to the original problem – using the previous result, find the capacitance *c* per unit of axial length of transmission line between the line and the ground plane.

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3. The normalized wavefunction $\psi(\vec{r})$ of a particle of mass m is found to be:

$$\psi(\vec{r}) = A\left[\left(\sqrt{6}\frac{x}{r} + \sqrt{6}\frac{z}{r}\right)e^{-r} + \left(12\sqrt{5}\frac{z^2}{r^2}\right)e^{-2r}\right],$$

where A is a constant.

a. What are the possible values of L^2 that can be measured and the probabilities of measuring each? Here $\vec{L}$ is the orbital angular momentum.

b. What is $\langle L^2 \rangle$?

c. What are the possible values of L_z that can be measured and the probabilities of measuring each?

d. If L_z is measured and the result is 0, what is the state of the system immediately after the measurement?

e. For the original state $\psi(\vec{r})$, what is the probability of finding $2\hbar^2$ and 0 if L^2 and L_x are measured at the same time?

Some of the following may be useful:

$$\int_0^\infty e^{-bx^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{b}} \qquad \int_0^\infty xe^{-bx^2}dx = \frac{1}{2b} \qquad \int_0^\infty x^2e^{-bx^2}dx = \frac{\sqrt{\pi}}{4b^{3/2}}$$

$$\int_0^\infty x^n e^{-ax}dx = \frac{n!}{a^{n+1}} \qquad n \in I$$

$$J_x^{(1/2)} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad J_y^{(1/2)} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad J_z^{(1/2)} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$J_x^{(1)} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad J_y^{(1)} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad J_z^{(1)} = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$L_x = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \quad L_y = \frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \quad L_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

$$Y_0^0(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$$

$$Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta \quad Y_1^{\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}$$

$$Y_2^0(\theta,\phi) = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right) \quad Y_2^{\pm1}(\theta,\phi) = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\phi} \quad Y_2^{\pm2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\phi}$$

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4. Consider two localized spin-$\frac{1}{2}$ particles which interact via the Heisenberg exchange interaction described by the Hamiltonian

$$\mathcal{H} = J\mathbf{S}_1 \cdot \mathbf{S}_2.$$

Here J is a constant energy and $\mathbf{S}_i = (S_{ix}, S_{iy}, S_{iz})$ is $2/\hbar$ times the spin operator for the i^{th} particle. These spin operators obey the commutation relations

$$[S_{i\alpha}, S_{i\beta}] = i\epsilon_{\alpha\beta\gamma}S_{i\gamma} \quad , \quad [S_{1\alpha}, S_{2\beta}] = 0.$$

Here α, β, and γ range over 1, 2, 3, indicating the x, y, z Cartesian components and $\epsilon_{\alpha\beta\gamma}$ is the permutation symbol. The notation $|\sigma\sigma'\rangle$ denotes a state with z-components of spin of σ on site 1 and σ' on site 2. An up on site 1 and a down on site 2 would be represented as $|\uparrow\downarrow\rangle$.

(a) The total spin operator commutes with the Hamiltonian, so that we can classify the states in terms of their total spin. There is one spin singlet and one spin triplet state. Find the energies of these two different states and an explicit wavefunction for all four eigenstates. (Note that one is threefold degenerate).

(b) Add a magnetic field pointing in the z-direction to the system, so that the Hamiltonian becomes

$$\mathcal{H}(t) = J\mathbf{S}_1 \cdot \mathbf{S}_2 - g\mu_B B(t)(S_{1z} + S_{2z}).$$

Show that the states you found in part (a) (or linear combinations of the degenerate states that you found) are instantaneous eigenstates of the new Hamiltonian for arbitrary $B(t)$. Determine the different instantaneous eigenenergies.

(c) The time-dependent Schroedinger equation is

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \mathcal{H}(t)|\psi(t)\rangle.$$

Suppose $B(t)$ is equal to zero until $t = 0$, after which it satisfies $B(t) = B_0 t$. The system starts in the state $|\psi(0)\rangle = (|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle)/\sqrt{2}$ at time $t = 0$. Find the state for all positive times.

(d) Using your result from (c), determine the probability that the spin is up along the z-direction on site 1 as a function of time, and the probability that the spin is up on site 2 as a function of time.

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Part II: Tuesday, August 30, 2011, 1:00pm - 5:00pm

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1. The heat capacity of a glass apparently always contains a term linear in T, the absolute temperature. The general consensus appears to be that the source of this contribution to the heat capacity is a distribution of non-interacting two-level sites distributed throughout the glass. The underlying physical origin of the two-level structures is unclear, but it seems to be an intrinsic feature of disordered glassy structures and not the result of some special chemical circumstances. The purpose of this problem is to show that this model gives a heat capacity that depends linearly on T.

Consider a material that contains N sites or structures that, independently of their precise nature, serve as two-level systems for atoms or small groups of atoms. These two-level systems do not interact with one another. The two energies for the Kth two-level system are $\pm \Delta_K$, where

$$\Delta_K = \frac{K}{N} \Delta_0, \ for\ K = 1,2,\ \ldots\ ,N.$$

Here Δ_0 is a positive energy. All of the energy levels are non-degenerate.

a. Calculate the Helmholtz free energy of this collection of N two-level systems for $N >> 1$.

HINT: Since, for $N >> 1$, the values of Δ_K are very closely spaced, one can use the connection

$$\sum_K \Rightarrow C(N,\Delta_0) \int_0^{\Delta_0} d\Delta\ .$$

Be sure to explain this connection and obtain the correct expression for $C(N,\Delta_0)$

You may leave your result in terms of a one-dimensional integral.

b. Calculate C_V, the heat capacity at constant volume, of the system, for $N >> 1$.

You may leave your result in terms of a one-dimensional integral.

c. Show that for $N >> 1$ and $\Delta_0/k_BT >> 1$, $C_V \propto T$. Here k_B is the Boltzmann constant and T is the absolute temperature of the system.

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2. A rigid, isolated container of volume V is divided by a rigid, thermally insulating, removable wall into two parts of volume V_1 and V_2, where $V = V_1 + V_2$. The two parts are initially filled with an ideal gas, with $N_1 = N_2 = N$, $T_1 = T_2 = T$, and $P_1 \neq P_2$. The wall between the two parts is then removed and the contents of the two parts are allowed to mix.

For N molecules of this gas at temperature T and pressure P, the thermodynamic potential (or Gibbs free energy) is given by

$$G(P, T) = Nk_B T \ln P + N\chi(T), \tag{1}$$

where $\chi(T)$ is a function of temperature alone, and k_B is the Boltzmann constant.

a. Calculate the final pressure of the system and the change in entropy of the system. Express your answer in terms of N, P_1, and P_2.

b. Show that

$$N\frac{d\chi(T)}{dT} = -C_p \ln(T) + A, \tag{2}$$

where $C_P = C_V + Nk_B$, provided that the heat capacities for N particles at constant pressure and constant volume (C_P and C_V) are independent of both temperature and pressure. The constant A is a constant of integration.

c. Now consider the case where initially $N_1 = N_2 = N$, $P_1 = P_2 = P$, and $T_1 \neq T_2$. Again the wall separating the two parts is removed and the contents are allowed to mix. Calculate the final temperature of the system and the change in entropy of the system. Express your answers in terms of N, C_P, T_1, and T_2.

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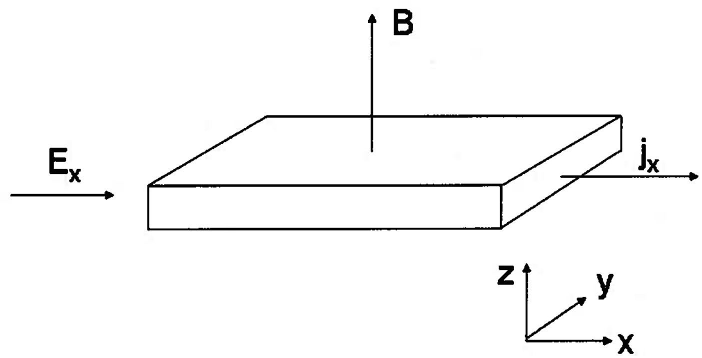
3. Concentration and effective mass of charge carriers are important characteristic parameters to study transport properties of a conductor. The Hall effect is widely used to measure the carrier concentration. When a longitudinal electric field E_x and a transverse magnetic field B are applied, the Hall field E_y develops in a direction perpendicular to B and E_x (see Figure below).



a) Assume that the charge carriers are electrons and write the equation of motion for the electrons in terms of their effective mass, the collision time τ and the electric and magnetic fields. Draw the vector corresponding to E_y in the figure, with the correct orientation. Express the Hall coefficient $R_H = E_y/ (j_xB)$ in terms of the electron density. What is the sign of the Hall coefficient?

b) Assume that the charge carriers are holes. What is the sign of R_H? Explain your answer.

c) Even though holes are treated like particles, they are vacant orbitals in the valence bands and the actual particles carrying the current are electrons in the valence band. If the charge carriers are electrons anyway, why is the sign of the Hall coefficient different for electrons in the valence band when compared to electrons in the conduction band?

d) The effective mass of charge carriers can be measured by cyclotron resonance when $\tau >> 1/\omega_C$, where ω_C is the cyclotron frequency.
   i. Assume that a constant magnetic field is applied in the z direction and find the expression of the cyclotron frequency from the equation of motion (neglect damping due to collisions).
   ii. To measure the cyclotron frequency (thus the effective mass) a RF electromagnetic field is applied in addition to the constant magnetic field. In which direction is the RF electric field applied and why?
   iii. Why is it important to have $\tau >> 1/\omega_C$? For a given conductor, what can you do to maximize τ in your experiment?

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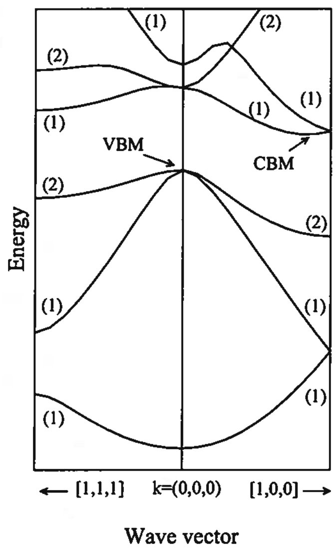
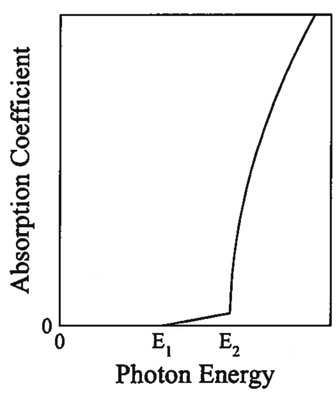
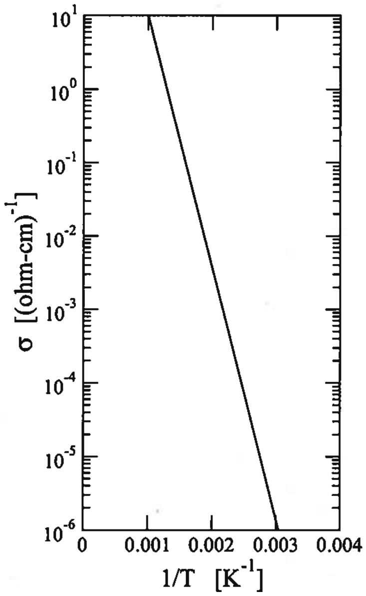
4. A hypothetical crystalline semiconducting material has a simple cubic Bravais lattice with lattice constant a. The plots below show the material's electronic band structure along selected directions in the Brillouin zone, its low-temperature optical absorption spectrum, and the temperature dependence of its electrical conductivity. In the band structure plot, the valence band maximum (VBM) and conduction band minimum (CBM) are marked, and the degeneracy of each band is noted in parentheses.







(a) Write down a set of primitive lattice vectors: $\vec{a_1}$, $\vec{a_2}$, and $\vec{a_3}$. Determine the corresponding primitive reciprocal lattice vectors: $\vec{b_1}$, $\vec{b_2}$, and $\vec{b_3}$. What is the shape and size of the Brillouin zone?

(b) Suppose the sample has dimensions $L_x = N_x a$, $L_y = N_y a$, and $L_z = N_z a$. Apply periodic boundary conditions to determine the set of allowed wave vectors, $\vec{k}$. How many allowed $\vec{k}$ vectors are there in the Brillouin zone?

(c) Determine the number of valence electrons per primitive cell. Explain your reasoning or show your work.

(d) How are the photon energies marked E_1 and E_2 on the absorption plot related to features of the electronic band structure? By considering the physical processes that can occur in each case, explain the three different regimes of absorption: no absorption when the photon energy is less than E_1, weak absorption when it is between E_1 and E_2, and strong absorption when it is greater than E_2. (Focus on the strength of absorption in the different regimes; do not worry about the detailed functional form of the absorption.)

(e) Describe the physics that gives rise to the functional form of $\sigma(T)$ shown in the conductivity plot.

(f) Estimate the band gap (in eV) of this semiconducting material. Show your work.

Possibly useful information:

$\hbar = 1.05 \times 10^{-34}$ J-s; $k_B = 1.38 \times 10^{-23}$ J/K; 1 eV $= 1.60 \times 10^{-19}$ J; $c = 3.00 \times 10^{8}$ m/s