Ph.D. Comprehensive Exam

Department of Physics
Georgetown University

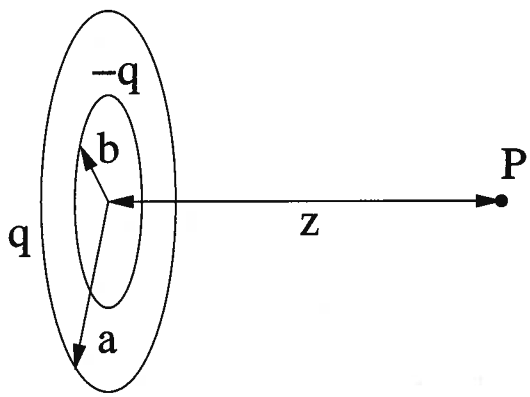
Part I: Monday, August 27, 2007, 2:00pm - 6:00pm

Instructions:

- This is a closed-book, closed-notes exam. A calculator may be used for mathematical computations but not for storing formulae.
- Each problem is worth 20 points.
- You should submit work for all of the problems. Note that in some cases, even if you get stuck on one part of a problem, you may be able to make progress on subsequent parts.
- If you need additional paper to work the problems, please use separate sheets for different problems.
- Show all your work.

Name: ______________________________

1. A uniformly charged ring with a total charge q and radius a is concentric and coplanar with another ring having charge $-q$ and radius b, where $a > b$, as depicted in the figure. Point P lies a distance z from the center of the ring, on the axis perpendicular to the plane of the rings.



(a) Determine the electric field $\vec{E}$ at point P.

(b) Determine the electric potential V at point P.

(c) Simplify the potential for the case of $z >> a, b$, and explain why your result makes sense.

Name:

2. In this problem you will show that the equilibrium number, n, of point defects in a perfect lattice is given by $n = Ne^{-\Delta U/k_BT}$, where N is the number of lattice sites and ΔU is the increase in energy associated with the creation of one defect.

(a) Show that the number of distinct ways of arranging n defects on N lattice sites is given by the binomial distribution, $W = \frac{N!}{n!(N-n)!}$. (Each lattice site can contain at most one defect.)

(b) Show that the free energy for this system at relatively low temperatures is approximately $F = F_o + n\Delta U + k_BTn \ln n/N$, where F_o is the free energy of the perfect lattice. (You will need Stirling's approximation: if $m \gg 1, \ln m! \approx m \ln m - m$. You can also use the fact that at 'relatively low' temperatures, the number of defects is much less than the number of lattice sites. For a typical solid at room temperature, $n \sim 10^{13} - 10^{18}$ per mole.)

(c) Use the free energy from the previous part to show that in equilibrium the number of defects is given by $n = Ne^{-\Delta U/k_BT}$.

Name:

3. Consider a two-dimensional lattice of atoms in which atomic displacements are restricted to be within the plane of the lattice. The lattice extends over an area of size $L \times L$. We will apply the Debye model to this system. Recall that the Debye approximation assumes an isotropic linear phonon dispersion relation with a cutoff at the Debye frequency, ω_D.

(a) Is the Debye model more appropriate for describing acoustic phonons, optic phonons, or both? How many acoustic branches does this system have?

(b) Apply the Debye model to this system in order to determine the density of phonon states, $D(\omega)$, as a function of frequency ω.

(c) Write down an expression for the total phonon energy at temperature T. You may leave dimensionless integrals in your answer.

(d) Determine the temperature dependence of the phonon heat capacity in the low-temperature regime ($kT << \hbar\omega_D$).

Name:

4. A particle of mass m is in a harmonic oscillator potential $V_a(x) = \frac{1}{2}k_a x^2$. The energy of the system is measured at time $t = 0$ and found to be $\hbar\omega_a/2$, where $\omega_a = \sqrt{k_a/m}$. Right after the energy measurement, the potential is suddenly changed to $V_b(x) = \frac{1}{2}k_b x^2$, with $k_b = k_a/4$, leaving the wavefunction momentarily undisturbed.

(a) If the energy is measured right after the potential is changed,

i. what is the probability that the particle will be in the ground state $|\phi_0\rangle$ of the new potential?

ii. what is the probability that the particle will be in the first excited state $|\phi_1\rangle$ of the new potential?

(b) To a good approximation, we can assume the remaining probability corresponds to finding the particle in the second excited state $|\phi_2\rangle$ of the new potential. Let $|\Psi(0)\rangle$ be the state of the system immediately after the potential is changed. Express $|\Psi(0)\rangle$ in terms of $|\phi_0\rangle$, $|\phi_1\rangle$, and $|\phi_2\rangle$. (You may assume all the coefficients are real and non-negative.)

(c) Determine $|\Psi(t)\rangle$, the state of the system a time t after the potential is changed.

(d) What is the expectation value of the energy at time t? (You may leave your answer in terms of $\omega_b = \sqrt{k_b/m}$.)

Harmonic Oscillator wavefunctions in position representation:

$$\phi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\phi_1(x) = \left[\frac{4}{\pi}\left(\frac{m\omega}{\hbar}\right)^3\right]^{1/4} x e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\phi_2(x) = \left(\frac{m\omega}{4\pi\hbar}\right)^{1/4}\left[2\frac{m\omega}{\hbar}x^2 - 1\right] e^{-\frac{m\omega}{2\hbar}x^2}$$

A few integrals that may be useful:

$$\int_0^\infty e^{-bx^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{b}}$$

$$\int_0^\infty x e^{-bx^2}\,dx = \frac{1}{2b}$$

$$\int_0^\infty x^2 e^{-bx^2}\,dx = \frac{\sqrt{\pi}}{4b^{3/2}}$$

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Part II: Tuesday, August 28, 2007, 2:00pm – 6:00pm

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- Each problem is worth 20 points.
- You should submit work for all of the problems. Note that in some cases, even if you get stuck on one part of a problem, you may be able to make progress on subsequent parts.
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- Show all your work.

Name____

1. A particle of mass m is constrained to move on the surface of a sphere of radius R; otherwise the particle is free.

(a) Show that the Hamiltonian of the system is given by $H = L^2/(2mR^2)$, where L is angular momentum. Which, if any, angular momentum observables are compatible with the Hamiltonian?

(b) Find the energy eigenvalues and the eigenvalues of all other compatible observables.

(c) Briefly describe the simultaneous eigenfunctions.

(d) What are the degeneracies of the lowest three energy levels?

(e) Now suppose there is a small uniform electric field in the z direction, so that the Hamiltonian includes a potential energy term of the form $V = -qEz$, where q is the charge of the particle and E is the magnitude of the field. Treating the potential as a perturbation, determine the first-order correction to the energy of the ground state.

(f) Return now to the case without an electric field. Suppose there are two identical non-interacting spin-1/2 particles constrained to the surface of the sphere. Write an expression for the ground-state of the two-particle system in terms of eigenstates of the single-particle Hamiltonian.

Name____

2. The energy dispersion relation of a solid near a valley of the conduction band can be approximated by the relation:

$$E(\vec{k}) = E_c + \frac{\hbar^2 k^2}{2m_c},$$

where m_c is the conduction band effective mass and E_c is the edge of the conduction band. L_x, L_y and L_z indicate the dimensions of the solid in the x, y, and z directions, respectively.

a) Assuming periodic boundary conditions, what are the allowed values for k_x, k_y, k_z?
b) What happens to the allowed energy levels when the dimension $L_z \ll L_x, L_y$? How does energy quantization lead to confinement of electrons in the x and y plane?
c) Estimate how small L_z needs to be in order to observe size quantization at room temperature for a solid like GaAs, with $m_c = 0.07\, m_e$.
d) Derive the expression for the density of states as a function of energy in the strictly two-dimensional case, $L_z = 0$, with the energy dispersion given above. Sketch a qualitative plot of the density of states as a function of $E' = E - E_c$.
e) Modify your sketch for the situation analyzed in parts b and c ($0 < L_z \ll L_x, L_y$).

Useful constants:
$k_B = 1.3806 \times 10^{-23}$ J/K
$h = 6.626 \times 10^{-34}$ J s
$m_e = 9.109 \times 10^{-31}$ kg

Name____

3. There is a mysterious region in free-space that contains a magnetic flux density, **B**, and an electric field, **E**, given by:

$$\mathbf{B} = 5.0 \times 10^{-4}\mathbf{a}_z \text{ T}$$

$$\mathbf{E} = 5.0\ \mathbf{a}_z \text{ V/m}$$

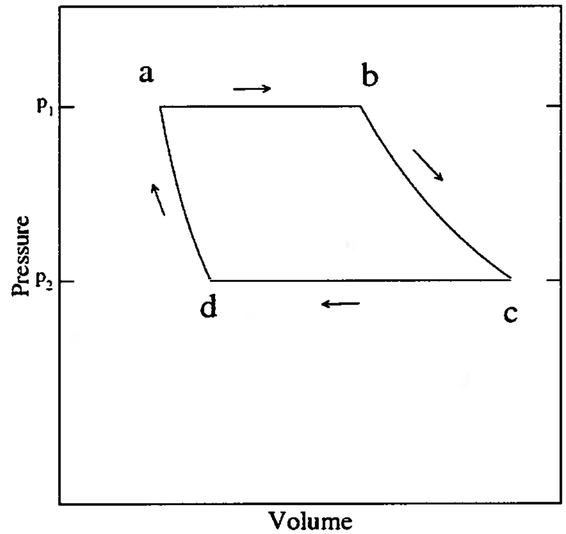
A proton, having a charge of $Q=1.602 \times 10^{-19}$ C and a mass of $m=1.673 \times 10^{-27}$ kg, enters the field-region at the origin with an initial velocity of $\mathbf{v}_o = 2.5 \times 10^5\mathbf{a}_x$ m/s.

(a) Describe the proton's motion in a qualitative manner, fully explaining your reasoning

(b) Calculate the proton's position after three complete cycles.

Name____

4. An ideal gas of N molecules, each with g degrees of freedom, is carried around the reversible closed cycle shown below. Two of the segments are adiabatic, and two are isobaric (constant pressure).



(a) What is the value of g for
  i. a monatomic gas?
  ii. a gas of diatomic molecules that can be treated as rigid rotors?

(b) Show that when an ideal gas undergoes an adiabatic process, pV^γ = constant, where γ is related to g. Determine the relation between γ and g. (Hint: use the ideal gas law, the internal energy of an ideal gas, and the first law of thermodynamics.)

(c) Of the states a, b, c, and d, which has the highest temperature? Which has the lowest temperature?

(d) For each segment of the loop, state whether the entropy of the gas is increasing, decreasing, or staying constant. Explain how you can tell.

(e) For each segment of the loop, state whether the gas is doing positive work, negative work, or no work.

(f) When cycled in this way, does the system act as a heat engine or a refrigerator? Explain.