Ph.D. Comprehensive Exam

Department of Physics
Georgetown University

Part I: Monday, July 20, 2015 , 1:00pm - 5:00pm

Instructions:

• This is a closed-book, closed-notes exam. The only electronic devices allowed are calculators provided by the department.

• Each problem is worth 50 points.

• You should submit work for all of the problems. In many cases, even if you get stuck on one part of a problem, you may be able to make progress on subsequent parts.

• Please work different problems on separate sheets of paper.

• Show all your work.

The parity operator $\mathbf{\Pi}$ acts on the eigenstates of the position and momentum operators $\mathbf{X}$ and $\mathbf{P}$ as follows:

$\mathbf{\Pi}\ |x> = |-x>$ and $\mathbf{\Pi}\ |p> = |-p>$.

1) Check if $\mathbf{X}$ and $\mathbf{P}$ commute or anticommute with $\mathbf{\Pi}$.
2) Consider the Hamiltonian operator $\mathbf{H} = V(\mathbf{X}) + \mathbf{P}^2/2m$. Use your result in 1) to find whether and under which conditions $\mathbf{H}$ and $\mathbf{\Pi}$ commute.
3) Consider the Hamiltonian operator corresponding to the potential
   $V(x) = V_0$ for $|x|< a$
   $V(x) = 0$ for $a \leq |x| \leq 2a$
   $V(x) = \infty$ for $|x| > 2a$.
   Plot the potential $V(x)$. Does $\mathbf{H}$ commute with $\mathbf{\Pi}$**?**
4) Sketch the wavefunctions of the two lowest energy eigenstates $|1>$ and $|2>$ (explain your reasoning for plotting them the way you do). Are these also eigenstates of $\mathbf{\Pi}$**?** Check this using the bra and ket formalism.
5) The two lowest energy eigenstates have different energy eigenvalues. Can you tell which one has higher energy and why, from the wavefunctions you drew (no need to calculate energy eigenvalues)? Is the linear combination of these two eigenstates $|+> = (|1> + |2>)/2^{1/2}$ also an eigenstate of $\mathbf{H}$?
6) Assume that a particle is in the state $|+>$ at the time $t=0$. Is the particle in a stationary state? (calculate the state at the time $t>0$ to check this)
7) Let V_0 go to infinity. How different are the wavefunctions of the two new lowest energy eigenstates $|1'>$ and $|2'>$ from the ones in 4? Plot the new wavefunctions for $|1'>$ and $|2'>$.
8) Is the linear combination of these two eigenstates $|+'> = (|1'> + |2'>)/2^{1/2}$ also an eigenstate of $\mathbf{H}$**?**
9) Assume that a particle is in the state $|+'>$ at the time $t = 0$. Is the particle in a stationary state? (calculate the state at the time $t>0$ to check this)
10) Is $|+'>$ an eigenstate of $\mathbf{\Pi}$? Explain how your answer is consistent with the commutation relation between $\mathbf{H}$ and $\mathbf{\Pi}$.

SYSTEM SPIN STATES FOR TWO AND THREE SPIN ½ FERMIONS

Consider a system consisting of two identical spin ½ fermions. Let $\vec{S}_1$ and $\vec{S}_2$ be the spin operators for particles 1 and 2, respectively. For the simultaneous eigenstates of $S_k^2 \equiv \vec{S}_k \cdot \vec{S}_k$ and $S_{kz} \equiv \hat{z} \cdot \vec{S}_k$, where $\hat{z}$ is the Cartesian unit vector in the z-direction and k = 1 or 2, please use the notation $|\tfrac{1}{2}, m\rangle_k$, where

$$S_k^2 |\tfrac{1}{2}, m\rangle_k = \frac{1}{2}\left(1 + \frac{1}{2}\right) \hbar^2 |\tfrac{1}{2}, m\rangle_k, \; S_{kz} |\tfrac{1}{2}, m\rangle_k = \hbar m |\tfrac{1}{2}, m\rangle_k, \; for\ k = 1\ or\ 2. \quad \textbf{(1)}$$

For the simultaneous eigenstates of S_1^2, S_2^2, S_{1z}, and S_{2z}, please use the notation

$$|m, m'\rangle_{1,2} = |\tfrac{1}{2}, m\rangle_1 \otimes |\tfrac{1}{2}, m'\rangle_2, \quad \textbf{(2)}$$

where ⊗ denotes the direct or Kronecker or tensor product. Of course, m and m′ take on the values ± ½.

a. Determine the eigenstates of S^2 and S_z, where

$$\vec{S} = \vec{S}_1 + \vec{S}_2, \quad \textbf{(3)}$$

and indicate the corresponding eigenvalues. For these eigenstates, please use the notation $|s, M\rangle$, where

$$S^2 |s, M\rangle = s(s+1)\hbar^2 |s, M\rangle, \; S_z |s, M\rangle = M\hbar |s, M\rangle. \quad \textbf{(4)}$$

Note carefully that

$|\tfrac{1}{2}, m\rangle_k$ are single-particle spin states,

$|m, m'\rangle_{1,2}$ are two-particle spin states,

and

$|s, M\rangle$ are two-particle spin states.

You may be more familiar with a notation given by

$$|\Uparrow\Uparrow\rangle = |\tfrac{1}{2}, \tfrac{1}{2}\rangle_{1,2}, \; |\Uparrow\Downarrow\rangle = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle_{1,2}, \; |\Downarrow\Uparrow\rangle = |-\tfrac{1}{2}, \tfrac{1}{2}\rangle_{1,2}, \; |\Downarrow\Downarrow\rangle = |-\tfrac{1}{2}, -\tfrac{1}{2}\rangle_{1,2}. \quad \textbf{(5)}$$

b. Consider a system consisting of three identical and indistinguishable non-interacting spin ½ fermions. Explain how the system spin states are grouped into singlets, doublets, etc. However, you need not work out the system spin states in terms of the single-particle spin states.

Solid State Physics Problem:

This problem concerns the three dimensional free electron gas model for metals. Let **m** denote the electron mass. Consider a system with electron number density **n**, and assume that there is no spin polarization.

1. Derive the relation between the Fermi wave vector $\mathbf{k_f}$ and the electron density **n**.

2. Calculate the average kinetic energy $\mathbf{E_K}$ per electron at $T = 0K$.

   Express E_K in terms of the Fermi energy defined as:

$$\epsilon_f = \frac{\hbar^2}{2m} k_f^2.$$

3. Qualitatively explain the exponent of the power of temperature, T, by which the specific heat of the metal scales for $k_B T \ll \epsilon_f$. How is it different from the specific heat of the ideal Boltzmann gas? Give an intuitive reason?
4. Estimate the average Coulomb interaction $\mathbf{E_c}$ among electrons in this metal.
5. Define the dimensionless ratio defined as $\mathbf{r_s = E_K/E_c}$ which provides a measure of the strength of Coulomb potential interaction energy to kinetic energy. Calculate this ratio for the metal. Simplify the expression to show how it is a function of the mean electron separation d $\equiv n^{-1/3}$, and the Bohr radius $a_0 = 5.29177 \times 10^{-11}$ meter $= \hbar^2/me^2$.
6. Calculate r_s for Copper (n = 8.5×10^{28} /m^3 and discuss if that metal is in the weak interaction regime.

7. Detail an experiment by which you could measure the Fermi energy of copper's electrons.

Lattice vibrations and thermal properties of solids

Consider a layered solid in which each layer is a square lattice of N atoms with lattice spacing a and linear size L. The atom displacements are confined to each layer, with infinitely rigid coupling between the layers.

1) Compute the density of states $D_p(\omega)$ for a given polarization p of the phonons in one layer, using periodic boundary conditions.

   Hint: Having enumerated the wavevectors compatible with the periodic boundary conditions, compute D_p as $dn/d\omega$, where n is the number of modes of frequency ω, i.e., with wavevector amplitude $\leq k$.

2) Compute, using $D_p(\omega)$ obtained in 1) and the Debye approximation, the heat capacity of each layer of the solid from the phonon total energy in the layer

$$U = \sum_p \int D_p(\omega) \frac{\hbar\omega}{e^{\hbar\omega/k_B T} - 1} d\omega$$

   where the sum is intended over different polarizations.

   Hint: The Debye approximation corresponds to taking the velocity of sound v to be constant for each polarization and the dispersion relation to be $\omega = vk$, where k is the wavevector amplitude. It is acceptable, in the calculation, to consider the same contribution for each polarization and isolate the temperature dependence (i.e., you can leave the answer in terms of a numerical integral, without evaluating it).

3) Discuss the temperature dependence of the heat capacity in the limit of low temperatures.

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Comprehensive Exam problems
Statistical Mechanics,

1. Consider a system of N localized particles which can exist in a ground state of zero energy or an excited state with energy ϵ and degeneracy g.

   (a) Derive expressions for the average energy E, the entropy S, and the heat capacity C_V (in terms of N, ϵ, g, and T) .

   (a) Derive explicit expressions (in terms of N, epsilon, g, and T)

   (b) If $g = 3$, find the temperature if the energy is maintained at the following values:

      i. $E = \frac{1}{2}N\epsilon$
      ii. $E = \frac{3}{4}N\epsilon$
      iii. $E = \frac{9}{10}N\epsilon$

   Comment on your results. In particular, in each case, explain the connection between the occupation levels of the energy states and the calculated temperature.

   (c) For each energy above, what will happen if the system is placed in contact with a reservoir of temperature $T = \epsilon/k_B$?

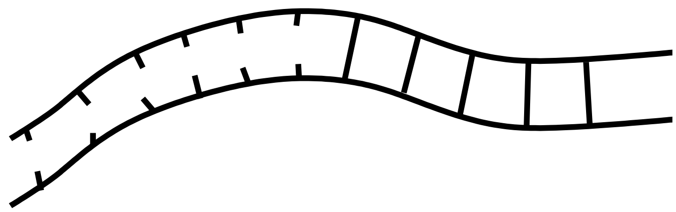
Statistical Mechanics

Stirling's Approximation $\quad \ln N! \simeq N \ln N - N$

Geometric series $\quad \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad \sum_{n=0}^{N} x^n = \frac{1-x^{(N+1)}}{1-x}$

Double stranded DNA can separate ('denature') by breaking base pairs, like a zipper. Consider a DNA molecule of N base pairs, each of which costs an energy Δ to open.

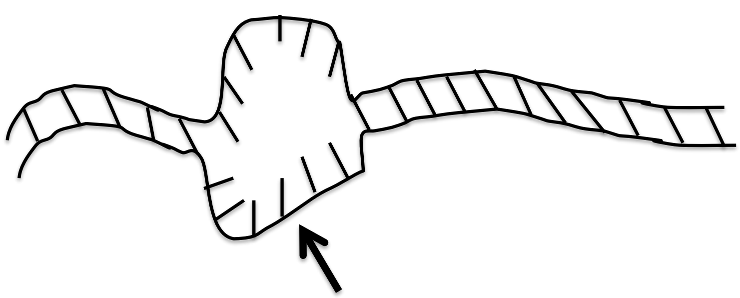
In a simple model, n base pairs open from one end of the DNA, as shown below, and the separated section is assumed to be stiff and keep its shape:



(a). Write down the general expression for the partition function $\mathcal{Z}$ for the *entire molecule*. Evaluate all sums or integrals to obtain a function $\mathcal{Z}(N, \Delta, k_B T)$.

(b). Show that the mean energy $\langle E \rangle$ and the mean number of open links $\langle n \rangle$ can be written in terms of a derivative of the partition function, and calculate $\langle E \rangle$.

DNA more often denatures by forming a *fluctuating bubble* in the middle, rather than opening a stiff section at the end:



n broken pairs out of N total pairs

The statistical weight of a loop of n broken base pairs is given by $\Omega(2n) = 2^{(2n)} A/(2n)^{3/2}$, where A is a constant. This accounts for all the different shapes of the loop.

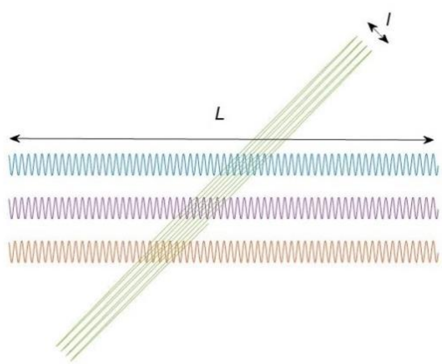
(c). Construct the partition function $\mathcal{Z}_{bubble}(n; N)$ of a *single bubble* formed by breaking n base pairs. Make sure you account for all places in which the bubble can form.

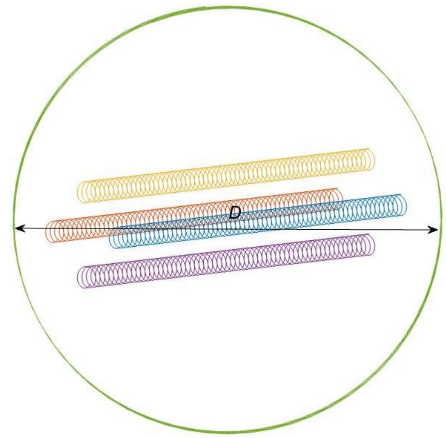
(d). Use $\mathcal{Z}_{bubble}(n; N)$ to find the free energy of the bubble. State the condition that the free energy of an equilibrium state satisfies.

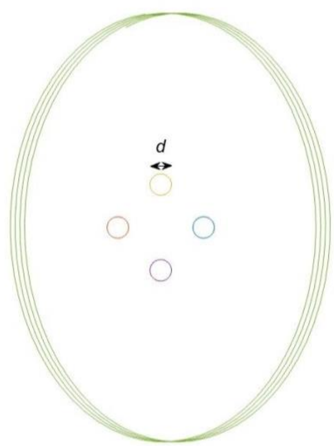
(e). Show that the equilibrium bubble size n^* is given by

$$\frac{\Delta}{k_B T} - 2 \ln 2 + \frac{3}{2n^*} + \frac{1}{N - 1 - n^*} = 0.$$

The diagrams show different views of one large coil with N turns, diameter D and length l, which surrounds four smaller coils with diameter d, n turns and lengths L. Assume that l<<L. The central axis of the large coil is at an angle of 45° with respect to that of the smaller coils. The smaller coils are located at a distance D/8 from the center of the large coil, and equidistant from each other. Be sure to state any assumptions you are using in your solutions. If you know it, you may use the formula for the B field in a solenoid without deriving it.







1. Find mutual inductance of the large coil with one of the small coils in terms of the given parameters.
2. Each of the smaller coils now has a current $I_S(t)=I_0\cos(\omega t)$ running through it. Find the emf in the larger coil.
3. If the wire of the larger coil has resistivity ρ and cross sectional area A, find the average (rms) power dissipated in it (assume it is a closed loop).

1. The Maxwell's equations for magnetostatics begin with

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = \mathbf{0}, \qquad \boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$$

where **B** is the magnetic field, **J** is the current density, and μ_0 is the magnetic permittivity. These equations give

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}$$

where $\boldsymbol{J} = \rho \boldsymbol{v}$ with ρ the charge density and **v** the velocity. Here, $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ and we also have $\boldsymbol{\nabla} \cdot \boldsymbol{A} = \mathbf{0}$.

The general solution is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} d^3 r'$$

Which means

$$A_i(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{J_i(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} d^3 r'$$

with *i = x, y, or z*. This result generalizes for a surface current to

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{K}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} d^3 r'$$

where the surface current density is **K**=σ**v**.

Your task is to find the vector potential of a spinning spherical shell of charge with a radius R, carrying a uniform surface charge density σ, and spinning at a constant angular velocity **ω**.

For easier integration, use coordinates in the figure below, with the z-axis along **r** and the angular velocity vector being **ω** = $\mathbf{e}_x$ ω sin ψ + $\mathbf{e}_z$ ω cos ψ.

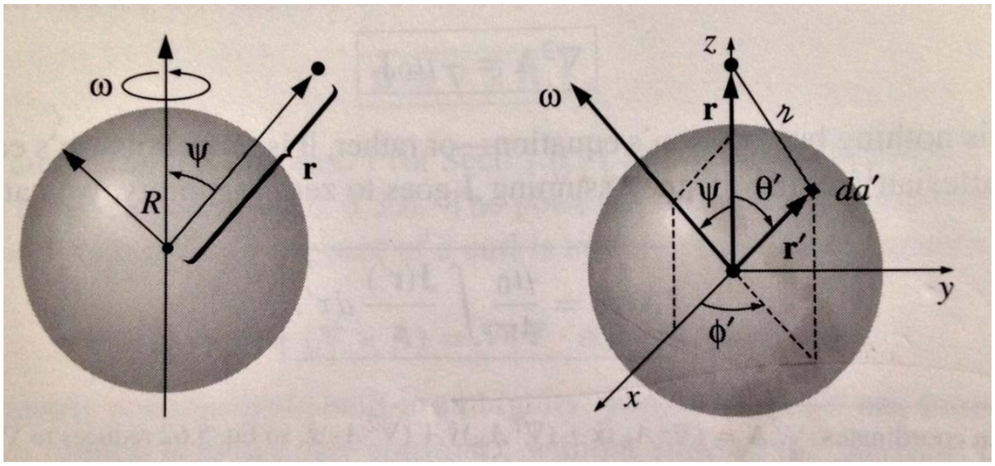


Figure showing the coordinate system to be used for the integration on the right.

Show that:

$$A_x(\boldsymbol{r}) = 0,\ \ A_z(r) = 0,\ \ A_y(\boldsymbol{r}) = \begin{cases} -\dfrac{\mu_0 R\sigma\omega r\, sin\,\psi}{3} & \text{for } r \le R \\ -\dfrac{\mu_0 R^4 \sigma\omega r\, sin\,\psi}{3r^3} & \text{for } r \ \ge R \end{cases}$$

or

$$\boldsymbol{A}(\boldsymbol{r}) = \begin{cases} \dfrac{\mu_0 R\sigma}{3}(\boldsymbol{\omega}\times\boldsymbol{r}) \ \ \text{for} \ \ r \le R \\ \dfrac{\mu_0 R^4\sigma}{3r^3}(\boldsymbol{\omega}\times\boldsymbol{r}) \ \ \text{for} \ \ r \ge R \end{cases}$$

2. Let $\boldsymbol{\omega}$ be along the z-direction $\boldsymbol{\omega} = \omega\mathbf{e_z}$ which yields

$$\boldsymbol{A}(r,\theta,\varphi) = \begin{cases} \dfrac{\mu_0 R\omega\sigma}{3} r\sin\theta\ \boldsymbol{e_\varphi} \text{ for } r \ \le R \\ \dfrac{\mu_0 R^4\omega\sigma}{3r^2}\sin\theta\ \boldsymbol{e_\varphi} \ \text{for} \ r \ \ge R \end{cases}$$

and

$$\boldsymbol{B}(r,\theta,\varphi) = \begin{cases} \dfrac{2\mu_0\sigma\omega R}{3}\,(\cos\theta\,\boldsymbol{e_r} - \sin\theta\,\boldsymbol{e_\theta}) & \text{for } r \le R \\ \dfrac{\mu_0\sigma\omega R^4}{3r^3}(2\cos\theta\,\boldsymbol{e_r} + \sin\theta\,\boldsymbol{e_\theta}) & \text{for } r \ge R \end{cases}$$

What are the boundary conditions for both **B** and **A** for spherical surface?