Ph.D. Comprehensive Exam

Department of Physics
Georgetown University

Part I: Monday, August 31, 2009, 1:00pm - 5:00pm

Instructions:

- This is a closed-book, closed-notes exam. Calculators are not allowed.
- Each problem is worth 40 points.
- You should submit work for all of the problems. Note that in some cases, even if you get stuck on one part of a problem, you may be able to make progress on subsequent parts.
- Please work different problems on separate sheets of paper.
- Show all your work.

Name: ______________________________

1. Consider a spin-1 nuclear particle with a magnetic dipole moment $\vec{\mu} = g\mu_N \vec{S}/\hbar$ in a uniform magnetic field $\vec{B} = (0, 0, B)$. The Hamiltonian is given by

$$H = -\vec{\mu} \cdot \vec{B}.$$

   (a) Construct the spin-1 matrices S_x, S_y, and S_z in the eigenbasis of S_z.
   Hint: Use the raising and lowering operators $S_+ = S_x + iS_y$ and $S_- = S_x - iS_y$. The effect of these operators on an eigenstate $|s, m\rangle$ of S_z is

$$S_+|s, m\rangle = \hbar\sqrt{(s-m)(s+m+1)}|s, m+1\rangle,$$

$$S_-|s, m\rangle = \hbar\sqrt{(s+m)(s-m+1)}|s, m-1\rangle.$$

   (b) At $t = 0$, the particle is in a normalized eigenstate of S_x that has eigenvalue $-\hbar$. Express this initial state, $\chi(0)$, in the S_z eigenbasis.

   (c) Find the state of the system, $\chi(t)$, at a later time.

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2. Consider a quantum particle moving in one dimension in a harmonic potential $V(x) = \frac{1}{2}kx^2$ with spring constant k and x the one-dimensional spatial coordinate.

The one-dimensional time-independent Schroedinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x),$$

with E the energy eigenvalue and m the particle mass.

(a) As you may recall, the ground-state wavefunction has the form $\psi_0(x) \propto e^{-bx^2}$. Determine the ground-state energy and the *normalized* ground-state wavefunction, with b expressed in terms of other constants specified in the problem. Make a sketch of $\psi_0(x)$. Some possibly useful integrals are given at the bottom of the page.

(b) The first excited state wavefunction has the form $\psi_1(x) \propto xe^{-bx^2}$. Determine the first excited state energy and the *normalized* wavefunction $\psi_1(x)$, with b expressed in terms of other constants. Make a sketch of $\psi_1(x)$.

The time-independent Schroedinger equation in spherical coordinates is

$$-\frac{\hbar^2}{2m}\nabla^2\psi(r,\theta,\phi) + V(r,\theta,\phi)\psi(r,\theta,\phi) = E\psi(r,\theta,\phi),$$

with the Laplacian given by

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right).$$

(c) Find the equation that a *spherically symmetric* wavefunction [*i.e,* $\psi(r,\theta,\phi) = \psi(r)$] satisfies if the potential is spherically symmetric. Express your answer concretely in terms of derivatives with respect to the spherical coordinates, and so on.

(d) Show that the spherically symmetric problem can be mapped onto an effective one-dimensional Schroedinger equation with an effective wavefunction $u(r)$ that is related to the spherically symmetric $\psi(r)$. What is the relationship between $u(r)$ and $\psi(r)$? What is the range of allowed values for r and what are the boundary conditions for $u(r)$?

(e) Solve your effective one-dimensional problem to find the *normalized* spherically symmetric wavefunction and eigenvalue for the ground state of the three-dimensional spherically symmetric simple harmonic oscillator $V(r) = \frac{1}{2}kr^2$. Note that you should be able to construct this answer with very little extra computation from what you have already completed. Make sure to use the appropriate normalization condition.

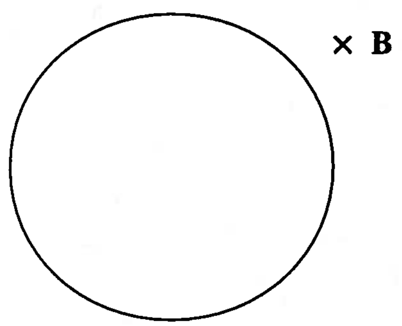
The following integrals may be useful:

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi} \qquad \int_{-\infty}^{\infty} x^2e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}$$

Name: ____

3. A circular loop of wire is placed in a constant magnetic induction as shown in the figure below (**B** pointing into the page). The loop initially has a radius r_0, but at t=0, the radius magically starts increasing at a constant rate k.

a. Find the expression for the electromotive force induced in the wire at time t.
b. If the wire has a fixed cross section A and a fixed resistivity, ρ, find its resistance as a function of time.
c. Using the results from a and b, find the induced current (magnitude and direction) in the loop.
d. Find the force per unit length needed to keep the coil magically expanding.



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4. Maxwell's Equations:

a) Fill in the complete set of Maxwell's equations **in differential form** in the presence of stationary charges and/or time-invariant current.

______________________________ (non-existence of magnetic monopoles)

______________________________ (Ampere's Law)

______________________________ (Gauss' Law)

______________________________ (Faraday's Law)

b) Charge in motion is simply an electric current, and because charge can never be created nor destroyed, a relationship between charge density and current density must always satisfy the condition given by:

______________________________ (Continuity Equation)

c) Ampere's Law (from above) should be modified to take into account the effects due to time-varying currents; this new expression led to the concept of "displacement current". State the modified Ampere's Law below, and show mathematically that the left-hand side of the expression is equal to the right-hand side (hint: you should know that the divergence of the curl of any vector function is always zero).

______________________________ (Ampere's Law for time-varying currents)

Proof:

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Part II: Tuesday, September 1, 2009, 2:00pm - 6:00pm

Instructions:

- This is a closed-book, closed-notes exam. Calculators may be used for mathematical computations but not for storing formulae.
- Each problem is worth 40 points.
- You should submit work for all of the problems. Note that in some cases, even if you get stuck on one part of a problem, you may be able to make progress on subsequent parts.
- Please work different problems on separate sheets of paper.
- Show all your work.

Name: ______________________________

1A. A small dust particle has a mass of about 10^{-8} g. It falls onto a glass of ice-cold water where it is supported by surface tension and moves freely in only two dimensions. (Note: $k_B = 1.38 \times 10^{-23} J/K$.)

(a) What is the average translational energy of the dust particle?

(b) What is the root-mean-squared instantaneous speed of its Brownian motion?

1B. Consider a hypothetical Fermi system with N particles in volume V and with the single-particle density of states $g(\epsilon)$ give by:

$$g(\epsilon) = \begin{cases} 0 & \text{for } \epsilon < 0 \\ \alpha V & \text{for } \epsilon > 0 \end{cases}, \tag{1}$$

where α is a constant.

For a system *at zero temperature*, find:

(a) the Fermi energy ϵ_F,

(b) the internal energy U,

(c) and the pressure P.

For a system *at non-zero temperature*:

(d) write an expression that could be used to determine the chemical potential $\mu(T)$. (You do not need to solve for $\mu(T)$, just indicate how it can be determined.)

Name:

2. Consider a system of N fixed non-interacting magnetic moments each of magnitude μ_0. Initially, the system is in equilibrium with a thermal reservoir at temperature T and is in a uniform external magnetic field of magnitude B. Each moment can be oriented only parallel or antiparallel to the magnetic field.

   (a) Determine the partition function of the system.

   (b) Determine the thermally-averaged energy E (or $\langle E \rangle$) of the system.

   (c) Show that your result for the energy E makes sense in the following limits:

      i. $\mu_0 B >> kT$.

      ii. $\mu_0 B << kT$.

   (d) Show that the entropy of the system is a function of B/T (i.e., not a function of these two variables separately). (Hint: The Helmholtz free energy is given by $F = E - TS = -kT \ln Z$.)

   (e) Now the system is thermally isolated from the reservoir. If the magnetic field is reduced when the system is no longer in thermal contact with the reservoir, what happens to the entropy of the system? What happens to the temperature of the system? Explain your reasoning.

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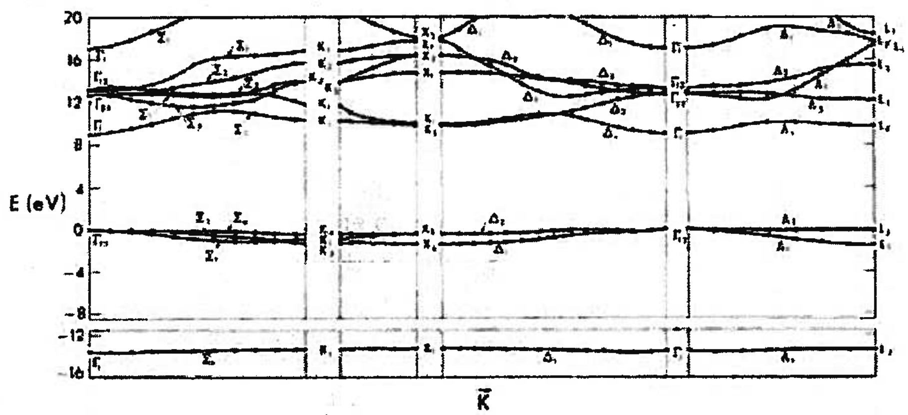
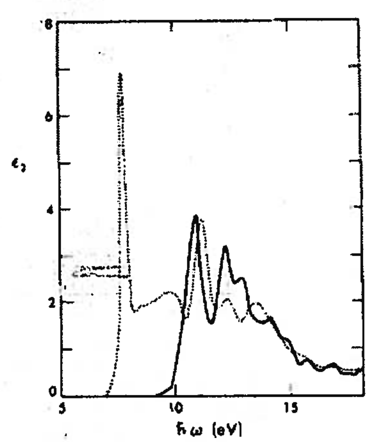
3. Crystals can be classified as metals, allowing for good conductivity of electric charges, and as insulators, where conductivity of charges is suppressed. A distinction between metals and insulators can be made through the electronic distribution in momentum space. Further distinctions can be drawn on the basis of the spatial electronic distribution, allowing for four main classes of crystals, knows as: (i) molecular crystals, (ii) ionic crystals, (iii) covalent crystals, and (iv) metallic crystals.

   (a) Figure 1 displays the calculated band structure of our crystal of interest ($E = 0$ marks the Fermi energy). Based on this electronic distribution in momentum space, is the crystal a metal or an insulator? Explain.



   (b) Valuable information about the band structure of crystals can be obtained by optical absorption. Figure 2 displays the imaginary part of the frequency-dependent dielectric function ϵ_2 (proportional to the absorption coefficient). Based on the results presented in Fig. 1, identify which of the two curves in Fig. 2 corresponds to a correct calculation of ϵ_2. Explain.



(Problem 3, continued)

In real space, the total lattice energy of our crystal, composed of N atoms (ions), can be very well described by the expression

$$U = 2N\left(Ae^{-r/R_0} - \frac{B}{r}\right), \tag{2}$$

where A, B, and R_0 are positive constants that depend on the lattice geometry and the kind atoms (ions) involved. Finally, r is the nearest-neighbor separation in the crystal.

(c) Explain the origin of the two terms in Eq. (2).

(d) Based on Eq. (2), in which of the classes (i) through (iv) mentioned before does the crystal belong? Explain.

(e) If you had the values of A, B, R_0, and the equilibrium value of r for our particular crystal, and plugged them into Eq. (2), would the resulting value of U be positive, negative, or zero? Explain.

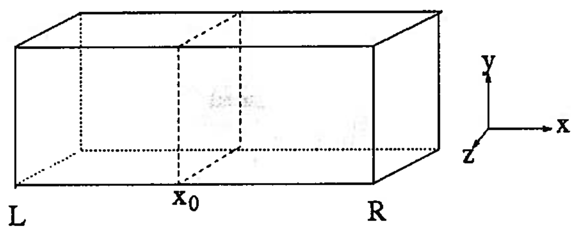
(f) Prove your answer in (e). All you need to know is that $R_0 < r$.

Name: ______________________________

4. Consider a gas of particles in a box. The left and right sides of the box are kept at different temperatures. The thermal conductivity κ of the gas is defined by:

$$j = -\kappa \, dT/dx$$

where j is the thermal energy transmitted across unit area per unit time and dT/dx is the temperature gradient.



(a) Why is there a minus sign in this expression?

(b) Use the kinetic theory of gases to derive the following expression for the thermal conductivity:

$$\kappa = 1/3 \quad (Cvl),$$

where C is the heat capacity of the gas, $v = \langle |\vec{v}| \rangle$ is the average speed of the particles in the gas (the average velocity is zero), and l is their mean free path. Assume the density of particles, n, is uniform, but the average energy of particles at x, $E(x)$, depends on the local temperature $T(x)$.
Hint: Roughly how many particles cross the $x=x_0$ plane per unit area per unit time traveling from left to right? What is the average energy of these particles, based on where they experienced their last collision? Repeat for particles that cross the plane coming from the right.

(c) The thermal conductivity of a solid can be modeled with the kinetic theory result. Suppose the solid **is not** a good electrical conductor. What are the "particles" that transport thermal energy? What physical quantities do v and l correspond to in this case?

(d) Does the heat capacity of the "particles" in (c) depend on temperature? Explain by considering the temperature dependence of the energy associated with these particles.

(e) Now suppose the solid **is** a good electrical conductor. How would your answers to (c) change for this case?