

Ph.D. Comprehensive Exam

Department of Physics
Georgetown University

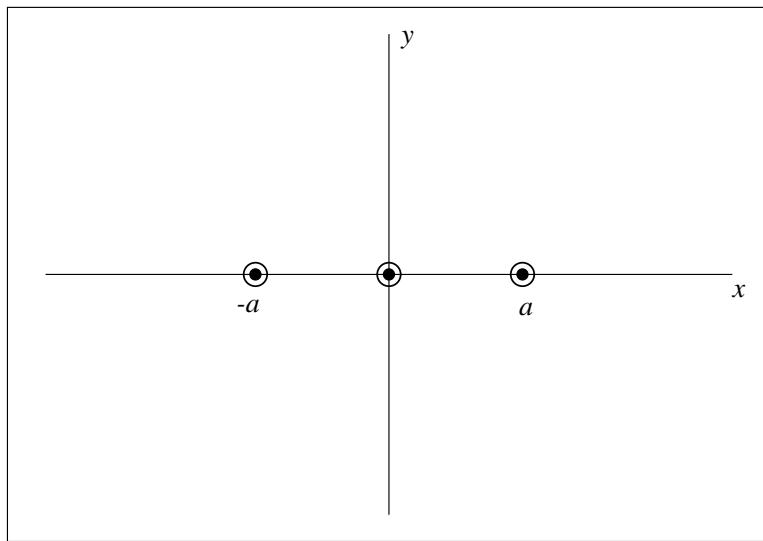
Part I: Monday, July 16, 2012, 1:00pm - 5:00pm

Instructions:

- This is a closed-book, closed-notes exam. The only electronic devices allowed are calculators provided by the department.
- Each problem is worth 50 points.
- You should submit work for all of the problems. In many cases, even if you get stuck on one part of a problem, you may be able to make progress on subsequent parts.
- Please work different problems on separate sheets of paper.
- Show all your work.

NAME: _____

1. Consider three straight, infinitely long, equally spaced wires (with radius $\ll a$), each carrying a constant current I_0 in the $+\hat{z}$ direction. Let the coordinates of the wires in the (x, y) plane be $(-a, 0)$, $(0, 0)$ and $(+a, 0)$, as shown in the picture. (The $+\hat{z}$ direction is out of the page.) The linear mass density of each wire is λ .



- (a) There are two points in the (x, y) plane where the magnetic field created by the three wires is zero. Determine the coordinates of these points.
- (b) In the picture above, make a **rough** sketch of the magnetic field line pattern. Briefly explain your reasoning.
- (c) Suppose that the middle wire is rigidly displaced upward by a very small distance $\epsilon \ll a$ to the point $(x, y) = (0, \epsilon)$, while the other two wires are held fixed. Describe qualitatively the subsequent motion of the middle wire. If it is oscillatory, find the frequency of small oscillations.
- (d) Suppose that the middle wire is instead rigidly displaced to the right by a very small distance $\epsilon \ll a$ to the point $(x, y) = (\epsilon, 0)$, while the other two wires are held fixed. Describe qualitatively the subsequent motion of the middle wire. If it is oscillatory, find the frequency of small oscillations.

NAME:

2. Determine if each of the following expressions for electric field can be an electromagnetic wave representation in free space. If not, explain why. If so, show that it is consistent with Maxwell's equations (and specify other conditions, if any, that must be satisfied), find the corresponding magnetic field $\vec{\mathbf{B}}(x, y, z, t)$, and determine the direction of wave propagation. Here c represents the speed of light in a vacuum, and E_0 , α , β , k , ω , and x_0 are constants. Assume a right-handed coordinate system.

(a) $\vec{\mathbf{E}}(x, y, z, t) = E_0 \sin[\alpha(x - ct)] \hat{\mathbf{x}}$

(b) $\vec{\mathbf{E}}(x, y, z, t) = E_0 e^{-\beta(y-ct)^2} \hat{\mathbf{z}}$

(c) $\vec{\mathbf{E}}(x, y, z, t) = E_0 \cos(kz - \omega t) \hat{\mathbf{x}}$

(d) $\vec{\mathbf{E}}(x, y, z, t) = \frac{E_0 x_0^2}{[x_0^2 + (x + ct)^2]} \hat{\mathbf{y}}$

NAME:

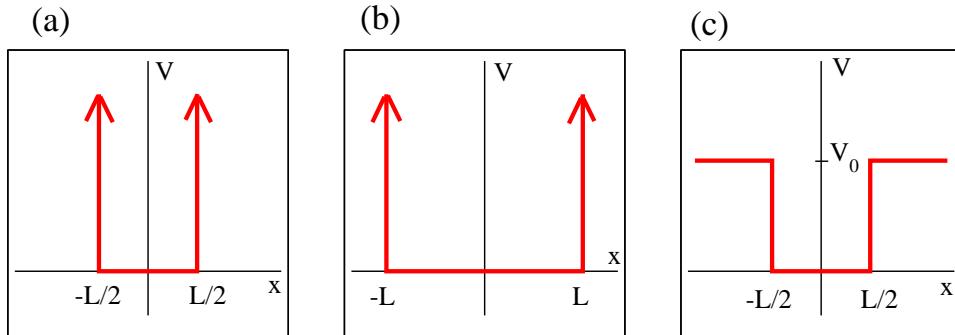
3. Consider a particle of mass m in an infinite square quantum well of width L :

$$V(x) = \begin{cases} 0 & \text{if } |x| < L/2 \\ \infty & \text{otherwise} \end{cases}$$

- (a) Derive the (normalized) eigenfunctions and energies of the ground state and the first excited state. Sketch these two eigenfunctions.
- (b) Suppose a particle is in the ground state of this well. The width of the well is then suddenly expanded symmetrically to twice the original width ($2L$), leaving the particle's wave function undisturbed.
 - * Write down an explicit expression (involving an integral) for the probability of finding the particle in the ground state of the new well of width $2L$. You do not need to evaluate the integral. Is the probability zero or non-zero? Explain how you can tell.
 - * Do the same for the probability of finding the particle in the first excited state of the new well of width $2L$.
- (c) Now consider the same particle in a finite square well of width L :

$$V(x) = \begin{cases} 0 & \text{if } |x| < L/2 \\ V_0 & \text{otherwise} \end{cases}$$

Would you expect the ground state energy of the particle in the finite well in Fig. (c) to be larger, smaller, or the same as that in the infinite well shown in Fig. (a)? Justify your answer conceptually (i.e., you don't need to solve the finite well problem in full.)



NAME: _____

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4. Consider a quantum mechanical system in a 2-dimensional Hilbert space. In the basis of energy eigenstates, the Hamiltonian H and an observable A are given by

$$H = \begin{pmatrix} \epsilon_0 & 0 \\ 0 & 3\epsilon_0 \end{pmatrix} \quad A = \begin{pmatrix} 1 & i\sqrt{2} \\ -i\sqrt{2} & 2 \end{pmatrix}.$$

- (a) Determine the eigenvalues and eigenstates of A .
- (b) At $t < 0$, the system is in the lowest energy state. (Assume that $\epsilon_0 > 0$.) At $t = 0$ a measurement of A is made. What are the possible outcomes of the measurement? Determine the probability of each outcome.
- (c) Assume the measurement in part (b) yields the lowest eigenvalue of A . At some time $t > 0$, a measurement of the energy is made. What are the possible outcomes? Determine the probability of each outcome.

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Part II: Tuesday, July 17, 2012, 1:00pm - 5:00pm

Instructions:

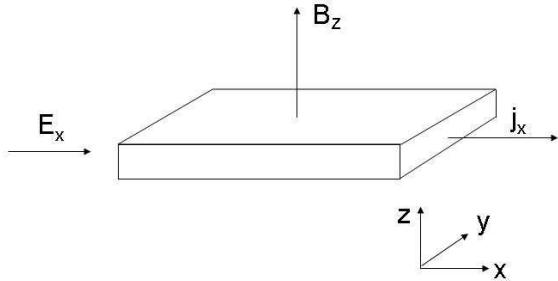
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NAME:

1. The crystal structure of solid lithium is body-centered cubic with a lattice constant of $a_0 = 3.50 \text{ \AA}$. (Note that a_0 is the length of the edge of the conventional cubic unit cell.) Lithium has one valence electron per atom. The Debye temperature, θ_D , of lithium is about 340 K. At room temperature, the electrical resistivity, ρ , of lithium is about $9.8 \times 10^{-8} \Omega\text{-m}$ in SI units (or $1.1 \times 10^{-17} \text{ s}$ in CGS units).
 - (a) Determine the electron number density n in bcc Li.
 - (b) Assuming a free-electron model with the electron density determined in part (a), estimate
 - * the Fermi wave number, k_F ,
 - * the Fermi energy, E_F ,
 - * the Fermi speed, v_F ,for electrons in Li.
 - (c) Using the information provided, obtain a rough estimate for the speed of sound in solid Li.
 - (d) Using the information provided, estimate the mean free path for an electron near the Fermi level in Li at room temperature.

NAME:

2. Consider a conducting slab sketched below. The charge carriers in the conductor are electrons with effective mass m^* , electric charge $-e$, and density n (number per unit volume). When a longitudinal static electric field E_x and a transverse magnetic field B_z are applied, the Hall field E_y develops in a direction perpendicular to B_z and E_x (see figure below).



- (a) The equation of motion for charge carriers can be written as

$$\left(\frac{d}{dt} + \frac{1}{\tau} \right) \vec{p} = \vec{F},$$

where τ is the collision time, \vec{p} is the carrier momentum averaged over a volume that is macroscopically small and microscopically large, and \vec{F} is the external force on a charge carrier. Show why the collision time can be introduced in the equation of motion in this form.

- (b) Show that in steady state, the equation of motion in (a) can be written in the form $\vec{J} = \hat{\sigma} \vec{E}$, where \vec{J} and \vec{E} are the current density and electric field vectors, respectively, and $\hat{\sigma}$ is a matrix with elements σ_{ik} such that $J_i = \sigma_{ik} E_k$ (where $i, k = x, y$, or z).
- (c) Show that in the limit $\omega_c \tau \gg 1$,

$$\sigma_{yx} = -\sigma_{xy} = \frac{nec}{B_z} \quad (\text{CGS units}) \quad \text{or} \quad \frac{ne}{B_z} \quad (\text{SI units}),$$

and

$$\sigma_{xx} = \sigma_{yy} \approx 0.$$

Note that ω_c is the cyclotron frequency, given by $\frac{eB}{m^* c}$ in CGS units or $\frac{eB}{m^*}$ in SI.

- (d) For a given conducting slab, which two experimental parameters could you vary to realize the condition $\omega_c \tau \gg 1$? Explain how these parameters affect the quantity $\omega_c \tau$.
- (e) If you could choose the material for the conductor, how would you select the material to better satisfy the condition $\omega_c \tau \gg 1$?
- (f) In the two-dimensional limit, the surface current density is defined as $\vec{J}_s = \vec{J}d$, where d is the (small) thickness of the conductor and the surface electron density is defined as $n_s = nd$. Express the Hall resistance, the ratio between the voltage drop across the y direction and the current along the x direction, $R_H = \frac{V_y}{I_x}$, in terms of n_s and B_z .

NAME:

3. (a) Each molecule of the protein myoglobin has a single site where an oxygen molecule O_2 can bind. Therefore the myoglobin can be thought of as having two states, where the number m of bound oxygen molecules is either $m = 0$ (with energy = 0) or $m = 1$ (with energy $-\epsilon < 0$).

- i. Suppose this myoglobin molecule is in equilibrium with the oxygen in the atmosphere. Write down the grand partition function for the protein molecule in terms of the chemical potential μ and temperature T of the atmospheric oxygen, and any relevant constants.
- ii. Derive an expression for $\langle m \rangle$ (the mean number of oxygen molecules bound to one molecule of myoglobin) as a function of the oxygen pressure p and temperature T .

Hint: The atmospheric oxygen can be described as an ideal gas. If we neglect internal motion (vibration and rotation) of these diatomic molecules, the chemical potential may be written $\mu/k_B T = \ln(n/n_Q)$, where $n_Q = (Mk_B T / 2\pi\hbar^2)^{3/2}$, n is the number of molecules per unit volume, and M is the mass per molecule.

- iii. Make a rough sketch of the oxygen binding curve, $\langle m \rangle$ versus p , for constant temperature.
- (b) Hemoglobin is similar to myoglobin, except that one hemoglobin molecule can bind up to four O_2 molecules. Assume that the four binding sites are independent and distinguishable, and that a hemoglobin with m oxygen molecules bound to it has energy $-m\epsilon < 0$. Derive an expression for the probability of finding one and only one O_2 molecule bound to the hemoglobin molecule. You may leave your expression in terms of the chemical potential and temperature of the atmospheric oxygen.

NAME:

4. A cylindrical container of height h and cross-sectional area A is filled with a monoatomic, classical, ideal gas. The mass of each atom is m , and there are N total atoms in the container. Each atom feels the same gravitational attraction to the Earth given by mg . In equilibrium, these atoms, much like those in our atmosphere, will be distributed in a way that results in more atoms at the bottom than at the top of the container (law of atmospheres).

- (a) Calculate $n(z)$, the gas concentration as a function of height above the bottom of the container. Remember that $P = nk_B T$.

Make sure to calculate the concentration at the bottom of the container, $n(0)$, *i.e.* do not leave that as an unknown constant.

- (b) Discuss the character of the distribution of the gas atoms for the two limiting cases, $T \rightarrow \infty$ and $T \rightarrow 0$, where T is the temperature of the system.